

## ON FAMILIES IN DIFFERENTIAL GEOMETRY

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ABSTRACT. Families of objects appear in several contexts, like algebraic topology, theory of deformations, theoretical physics, etc. An unified coordinate-free algebraic framework for *families of geometrical quantities* is presented here, which allows one to work without introducing *ad hoc* spaces, by using the language of differential calculus over commutative algebras. An advantage of such an approach, based on the notion of *sliceable structures* on cylinders, is that the fundamental theorems of standard calculus are straightforwardly generalized to the context of families. As an example of that, we prove the *universal* homotopy formula.

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## CONTENTS

1. Introduction	2
2. Cylinders and families of geometrical quantities	4
3. Derivatives of families	5
4. Integration of families	7
5. Families of covariant quantities	9
6. Example of families of “controvariant quantities”: vector fields and derivations	11
7. Vertical and horizontal differential forms	12
8. Families of differential forms and their natural operations	15
9. The homotopy formula	17
Acknowledgements	19
References	19

TABLE 1. List of main symbols.

$M$	a smooth manifold
$\mathcal{P}$	the manifold of parameters
$\mathbb{I}$	the closed interval $[0, 1]$
$\iota_g$	the slicing map
$M_g$	the slice determined by $g \in \mathcal{P}$
$\Theta$	a geometrical quantity on $M \times \mathcal{P}$
$\Theta_g$	restriction of $\Theta$ to the slice $M_g$
$\{\Theta_g\}_{g \in \mathcal{P}}$	family determined by $\Theta$
$F^\circ(\pi)$	pull-back bundle
$\pi_M, \pi_{\mathcal{P}}$	canonical projections
$D$	functor of derivations
$D_\pi^v$	functor of $\pi$ -vertical derivations
$D_\pi$	functor of derivations along $\pi$
$\text{Diff}$	functor of differential operators
$\Gamma(\pi)$	module of sections of $\pi$
$\Gamma_c(\pi)$	submodule of compact-supported sections of $\pi$
$\tilde{X}$	canonical lift of $X$
$\nabla_X$	der-operator associated with $X$
$X(F)$	the derivative of a family of maps $F$
$F'$	infinitesimal homotopy
$\mathcal{I}_{\mathbb{R}}, \mathcal{I}_a^b$	integration operators
$\mathcal{F}$	a functional on $C^\infty(\mathcal{P})$
$Q^\vee$	the dual of module $Q$
$\pi^\vee$	the dual of bundle $\pi$
$\overline{A}$	smooth envelope of $A$
$\text{Spec}(A)$	spectrum of $A$
$P \tilde{\otimes} Q$	smoothened tensor product
$\tilde{\mathcal{F}}$	canonical lift of functional $\mathcal{F}$
$\mathbf{F}$	a functor of differential calculus
$\Phi$	representing object of $\mathbf{F}$
$\Phi_F$	$F$ -horizontal sub-module
$\Lambda_f^1$	$f$ -horizontal 1-forms
$\Lambda^{p,q}$	forms of type $(p, q)$
$p_{p,q}$	canonical projector
$\mathcal{I}_f$	ideal of $f$ -horizontal forms
$\overline{\Lambda}_f$	algebra of $f$ -vertical forms
$\overline{d}$	horizontal differential

## 1. INTRODUCTION

Instances of families of geometrical objects can be found in Differential Geometry, where they play an essential role in the proof of key theorems. They are relevant in Algebraic Geometry as well, but this is not touched upon here. The purpose of this paper is to provide a conceptual approach to the theory of *families of geometrical quantities*—a common denomination which unifies such definitions as families, deformations, homotopies, isotopies, motions, etc.<sup>1</sup> We ventured calling it “conceptual” mainly for two reasons. First, it makes self-evident such a property as smoothness, which, despite its elementar character, it is usually defined in a slightly cumbersome way. Second, it allows a rigorous and straightforward generalization

<sup>1</sup>It is left to the reader the task to particularize the theory to the cases of personal interest.

of important theorems of differential calculus to the context of families (see, on this concern, the universal homotopy formula, proved in Section 9).

Such an approach cannot be obtained without exploiting the *logic of differential calculus over commutative algebras*, a theory pioneered by A. M. Vinogradov in the seventies (the main ideas can be found in the papers [7, 6], while the book [4] provides an elementary introduction to the subject). Roughly speaking, this “logic” is composed of the so-called *functors of differential calculus* (e.g., differential operators, derivations, etc.), each of which is accompanied by its representative object (e.g., the module of jets, differential forms, etc.). It turns out that representative objects are themselves functors, but contravariant ones (unlike the functors of differential calculus, which are covariant ones). However, the main difference between functors and representative objects is that the former are absolute, while the latter are relative, i.e., they depend on the module category in which they are defined. For instance, in order to recover the familiar definition of differential forms over a smooth manifold, one has to introduce the category of geometric modules over smooth algebras. But not only differential forms, even the whole calculus over smooth manifolds constitutes a chapter of the logic of differential calculus over commutative algebras, the key being provided by the so-called *Spectral Theorem* (see [4]), an isomorphism between the category of smooth manifolds and the category of smooth algebras. In other words, the logic of differential calculus over commutative algebras allows to formalize any well-known notion of differential calculus over smooth manifolds in terms of objects, morphisms, endofunctors and their representative objects in the categories of smooth algebras and geometric modules. But, most importantly, it allows to *define* notions (from smoothness itself) and theorems (e.g., the Newton–Leibniz formula) of differential calculus in far more general contexts than smooth manifolds (see, for instance, [5]) and, in the present case, in the context of families.

To begin with, introduce the idea of a geometrical quantity  $\Theta_M$  on  $M$ . Informally speaking, symbol  $\Theta$  denotes the *kind* of the quantity  $\Theta_M$  (which may be a function, a map, or a section of a vector bundle), while index “ $M$ ” is a remainder of the fact that our quantity is associated with  $M$  (i.e., it is a function on  $M$ , a map from  $M$  to another manifold, or a vector bundle over  $M$ ). Let now  $q$  be a point of an auxiliary manifold  $\mathcal{P}$ , henceforth called the *manifold of parameters*. A set  $\{\Theta_q\}_{q \in \mathcal{P}}$  of geometrical quantities of kind  $\Theta$  on the manifold  $M$  is what is usually referred to as a *family* of geometrical quantities (of kind  $\Theta$ ) on  $M$ . However, even from a mere notational point of view,  $\{\Theta_q\}_{q \in \mathcal{P}}$  is not an happy choice, since the symbol  $q$ , which stands for a point of an extra manifold, is attached to the symbol  $\Theta$ , which denotes a geometrical quantity on  $M$ . Conceptually, such a notation immediately reveals its limits, since it is not even able to clarify the relationship between the smoothness of the whole family and the smoothness of each of its member. So, the first aim of this paper is to replace the naive idea

(1) geometrical quantity  $\Theta_M \mapsto$  family of geometrical quantities  $\{\Theta_q\}$

with a more conceptual one

(2) geometrical quantity  $\Theta_M \longmapsto$  geometrical quantity on the  $\Theta$   
cylinder  $M \times \mathcal{P}$  of the same kind as  $\Theta_M$ .

The reader should keep in mind that, throughout this paper, we retain the name *cylinder* for the cartesian product  $M \times \mathcal{P}$ , since one of the most common instances of families is obtained when the manifold of parameter is  $\mathbb{R}$ , or an interval in it.

In order to work in the logic of differential calculus over commutative algebras, the idea (2) must be adopted as the most fundamental one, since it express the idea of a family in terms of an *algebra extension*  $C^\infty(M) \mapsto C^\infty(M \times \mathcal{P})$ , while the usual

one (1) can be retained for more descriptive purposes. This change of perspective makes it straightforward to express, in a natural and easy way, such matters as smoothness, derivation (Section 3), integration with respect to a parameter (Sec. 4), of families of geometrical quantities, and other relevant properties (like that of being analytic, algebraic, meromorphic, etc.) which are not investigated here.

The second aim of this paper is to introduce and to systematically study what we called the *sliceable* quantities on cylinders, whose appearance in the theory of families is explained as follows. The passage from (2) to (1) inevitably requires the *slicing maps*  $\iota_q : p \in M \rightarrow (p, q) \in M \times \mathcal{P}$ . This leads to think that *any* geometrical quantity  $\Theta$  on the cylinder can be “sliced” into a family  $\{\Theta_q\}_q$ , where  $\Theta_q$  is obtained “by applying”  $\iota_q$  to  $\Theta$ , and the meaning of “applying” depends on the kind of  $\Theta$  (for instance, if  $\Theta$  is a map, then  $\Theta_q$  is its pull-back  $\iota_q^*(\Theta)$ ). Now one should notice that, first, not all quantities on the cylinder can be sliced and, second, that the “slicing operation” may have a nontrivial kernel. The first phenomenon is typical of contravariant quantities (like vector fields), while the second concerns covariant quantities (to which Section 5 is dedicated). Hence, we shall call *sliceable* those quantities  $\Theta$  on the cylinder for which it makes sense to consider the restriction  $\Theta|_{M \times \{q\}}$  to the image of  $\iota_q$  and such that the correspondence

$$(3) \quad \text{sliceable quantity } \Theta \text{ on } M \times \mathcal{P} \longleftrightarrow \text{family } \{\Theta|_{M \times \{q\}}\} \text{ on } M$$

becomes one-to-one. Unlike functions and maps, which are all sliceable, sliceable vector fields will be understood as the sub-functor of *vertical* derivations (Section 6) and, dually, sliceable differential forms will be their annihilator and, as such, called *horizontal* (Sections 7 and 8). This confirms the key role played by the geometry of cylinders in the conceptual study of families of quantities.

Finally, since the correspondence (3) suggests that “anything to which  $\iota_q$  can be applied” can be considered as a family, then  $\iota_q$  may be also understood as the bundle pull-back  $\iota_q^\circ$ . This leads to the notion of a *family along a map* (Section 2), which, among other things, allows to give the conceptual definition of an *infinitesimal homotopy*.

## 2. CYLINDERS AND FAMILIES OF GEOMETRICAL QUANTITIES

In this section we collect basic notations and definitions.

A product  $M \times \mathcal{P}$  is a *cylinder* over  $M$ . An element  $q \in \mathcal{P}$  is called a *parameter*. Map  $\iota_q : M \rightarrow M \times \mathcal{P}$ ,  $M \ni p \mapsto (p, q) \in M \times \mathcal{P}$  is the *slicing map* associated with  $q$ , and  $M_q \stackrel{\text{def}}{=} \iota_q(M) = M \times \{q\}$  is the *slice* of parameter  $q$ . Obviously,  $\iota_q$  is a smooth embedding. Its restriction, still denoted by  $\iota_q$ , is a diffeomorphism between  $M$  and  $M_q$ .

Let  $\Theta$  be a geometrical quantity on  $M$  such that it makes sense to consider its restriction  $\Theta_q \stackrel{\text{def}}{=} \Theta|_{M_q}$ , for all  $q \in \mathcal{P}$ . Then the set  $\{\Theta_q\}_{q \in \mathcal{P}}$  is made of geometrical objects of the same kind as  $\Theta$ .

**Remark 1.** *It should be stressed that each element of  $\{\Theta_q\}_{q \in \mathcal{P}}$  lives on a different manifold, namely, the slice  $M_q$ . Nonetheless,  $\iota_q$  allows to transport  $\Theta_q$  back to  $M$ . This way, an object  $\Theta_q$  on  $M$  is obtained.*

**Definition 1.** *A set  $\{\theta_q\}_{q \in \mathcal{P}}$  is called a (smooth) family of geometrical quantities if there exists a quantity  $\Theta$  on  $M \times \mathcal{P}$  such that  $\theta_q$  corresponds to  $\Theta_q$  via  $\iota_q$ , for any  $q \in \mathcal{P}$ .*

The advantage of Definition 1 is that a family  $\{\Theta_q\}_{q \in \mathcal{P}}$  is smooth by default, allowing us to skip the modifier “smooth” in the sequel. Depending on the kind of  $\Theta$ , Definition 1 can be specialized as follows. A *family of functions* on  $M$ , parametrized

by  $\mathcal{P}$ , is a function  $f \in C^\infty(M \times \mathcal{P})$ . Similarly, a *family of maps* from  $M$  to  $N$ , parametrized by  $\mathcal{P}$ , is a smooth map  $F : M \times \mathcal{P} \rightarrow N$ .

Definition 1 roughly says that a family is obtained by “slicing something which lives on the cylinder” (by means of the  $\iota_q$ ’s). So, Definition 1 can be extended if one introduces more general objects “which can be sliced”. Such objects may be sections of an induced vector bundle  $F^\circ(\pi)$ , where  $F : M \times \mathcal{P} \rightarrow N$  is a smooth map and  $\pi$  is a vector bundle on  $N$ .

**Definition 2.** *An element of the  $C^\infty(M \times \mathcal{P})$ -module  $\Gamma(F^\circ(\pi))$  is called a family of sections of  $\pi$  parametrized by  $\mathcal{P}$  along  $F$  (or just a family of sections of  $\pi$  parametrized by  $\mathcal{P}$  when  $M = N$  and  $F = \pi_M$ ).*

**Remark 2.** *In fact, any element  $\sigma \in \Gamma(F^\circ(\pi))$  can be sliced into a family  $\{\sigma_q\}_{q \in \mathcal{P}}$ , where  $\sigma_q \stackrel{\text{def}}{=} \iota_q^\circ(\sigma)$ , and assignment  $\sigma \mapsto \{\sigma_q\}_{q \in \mathcal{P}}$  is one-to-one. Accordingly, we call  $\sigma$  sliceable but, as it turns out, not all geometrical quantities on the cylinder will be sliceable.*

It is worth observing that Definition 1 is not a particular case of Definition 2, since families of maps cannot be seen as elements of a module.

**Remark 3.** *A section  $\sigma \in \Gamma(\pi_M^\circ(\pi))$  can be sliced into a family of sections  $\{\sigma_q\}_{q \in \mathcal{P}} \subseteq \Gamma(\pi)$ , since  $\pi_M \circ \iota_q = \text{id}_M$  for all  $q$ ’s. In general, if  $\sigma$  is a section of  $F^\circ(\pi)$ , then  $\sigma_q$  is not a section of  $\pi$ , but a section of  $\pi_q \stackrel{\text{def}}{=} F_q^\circ(\pi)$  instead, i.e.,  $\sigma_q \in \Gamma(\pi_q)$ ,  $q \in \mathcal{P}$ . Informally speaking,  $\sigma$  defines a family  $\{\sigma_q\}_{q \in \mathcal{P}}$  of sections of a family of vector bundles  $\{\pi_q\}_{q \in \mathcal{P}}$ .*

It is worth observing that  $\Gamma(\pi) \subseteq \Gamma(\pi_M^\circ(\pi))$  via the map  $\sigma \mapsto 1_{C^\infty(M \times \mathcal{P})} \otimes \sigma$ . The image of this embedding is constituted of sections of  $\pi_M^\circ(\pi)$  that do not depend on the extra parameter, and, as such, may be referred to as *constant*.

### 3. DERIVATIVES OF FAMILIES

Due to their straightforwardness, all proofs in this sections will be omitted. We also assume that the reader knows about vertical derivations, derivations along a map, related derivations, and the theory of smooth envelopes (see [4] for more details). In the sequel, both  $C^\infty(M)$  and  $C^\infty(\mathcal{P})$  are naturally understood as subalgebras of  $C^\infty(M \times \mathcal{P})$  via the canonical monomorphisms  $\pi_M^*$  and  $\pi_{\mathcal{P}}^*$ , respectively, and  $C^\infty(M) \otimes_{\mathbb{R}} C^\infty(\mathcal{P})$  as a subalgebra of  $C^\infty(M \times \mathcal{P})$ , via the product map  $\pi_M^* \otimes \pi_{\mathcal{P}}^*$ .  $C^\infty(M \times \mathcal{P})$  is tacitly understood both as a  $C^\infty(M)$ - and as a  $C^\infty(\mathcal{P})$ -module.

We show how the peculiar geometry of  $M \times \mathcal{P}$  allows to lift any vector field on  $\mathcal{P}$  to a  $\pi_M$ -vertical vector field. In its turn, such a lift allows to give an intrinsic definition of *derivative* of a family.

Let  $P$  be a  $C^\infty(M \times \mathcal{P})$ -module.

**Lemma 1.** *Given a  $P$ -valued derivation  $X$  (resp.,  $Y$ ) of  $C^\infty(M)$  (resp.,  $C^\infty(\mathcal{P})$ ), there exists a unique  $P$ -valued derivation  $Z$  of  $C^\infty(M \times \mathcal{P})$ , simultaneously extending  $X$  and  $Y$ .*

**Lemma 2.** *Given vector fields  $X \in D(M)$  and  $Y \in D(\mathcal{P})$ , a unique vector field  $Z \in D(M \times \mathcal{P})$  exists, such that*

$$\begin{aligned} Z \circ \pi_M^* &= \pi_M^* \circ X, \\ Z \circ \pi_{\mathcal{P}}^* &= \pi_{\mathcal{P}}^* \circ Y. \end{aligned}$$

The last two conditions mean that  $Z$  is  $\pi_M$ -related to  $X$  and  $\pi_{\mathcal{P}}$ -related to  $Y$ . Observe that if  $X = 0$  (resp.,  $Y = 0$ ) then  $Z$  is  $\pi_M$ -vertical (resp.,  $\pi_{\mathcal{P}}$ -vertical). In other words, it holds the following Corollary.

**Corollary 1.** *Any vector field  $X \in D(M)$  (resp.,  $Y \in D(\mathcal{P})$ ) can be lifted to an unique  $\pi_{\mathcal{P}}$ -vertical vector field  $\tilde{X}$  (resp.,  $\pi_M$ -vertical vector field  $\tilde{Y}$ ) of  $M \times \mathcal{P}$ .*

Vector field  $\tilde{X}$  (resp.,  $\tilde{Y}$ ) above is the *canonical lifting* of  $X$  (resp.,  $Y$ ).

**Remark 4** (Coordinates). *Let  $\{x^1, \dots, x^n\}$  be local coordinates on  $M$  and let  $\{y^1, \dots, y^m\}$  be local coordinates on  $\mathcal{P}$ . Then the lifting  $Z$  of  $X = X^i \frac{\partial}{\partial x^i}$  and  $Y = Y^j \frac{\partial}{\partial y^j}$  (see Lemma 1) is given by  $Z = \pi_M^*(X^i) \frac{\partial}{\partial x^i} + \pi_{\mathcal{P}}^*(Y^j) \frac{\partial}{\partial y^j}$ .*

Since  $\Gamma(\pi_M^\circ(\pi)) = C^\infty(M \times \mathcal{P}) \otimes_{C^\infty(M)} \Gamma(\pi)$ , we also have the next Proposition.

**Proposition 1.**  $\nabla_X \stackrel{\text{def}}{=} \tilde{X} \otimes \text{id}$  is a well-defined first-order differential operator on  $\Gamma(\pi_M^\circ(\pi))$ .

More precisely, operator  $\nabla_X$  from Proposition 1 is a *der-operator* (see [2]) in  $\Gamma(\pi_M^\circ(\pi))$  over  $\tilde{X}$ . It allows to extend the notion of derivative to smooth families.

**Definition 3.** *Given  $\sigma \in \Gamma(\pi_M^\circ(\pi))$  and  $X \in D(\mathcal{P})$ , the smooth family  $\nabla_X(\sigma)$  is called the derivative of  $\sigma$  with respect to  $X$ .*

Let  $F : M \times \mathcal{P} \rightarrow N$  be a family of maps, and suppose that  $q$  is running along the trajectory of a vector field  $X \in D(\mathcal{P})$ . Then the  $F_q$ 's describe a “trajectory” in some “space of maps”, i.e., what is usually called a *deformation*.<sup>2</sup> In the standard approach, one tries to add some differentiable structure to this “space of maps”, in order to make it possible to compute the “velocity” of the deformation. Thanks to Definition 2, the idea of velocity of a deformation can be formalized algebraically, without even thinking about the “space of maps”.

More precisely, consider the composition  $X(F) \stackrel{\text{def}}{=} \tilde{X} \circ F^*$ , which is a vector field along  $F$ , i.e., a smooth section of the bundle  $F^\circ(\tau_N)$  induced from the tangent bundle of  $N$  by  $F$ . According to Definition 2,  $X(F)$  represents a smooth family of vector fields parametrized by  $\mathcal{P}$  along  $F$ . Moreover, as anticipated by Remark 3, the member of the family  $X(F)$  which corresponds to the parameter  $q$  is the vector field along  $F_q$  given by

$$(4) \quad X(F)_q = \iota_q^* \circ \tilde{X} \circ F^*.$$

So, Definition 4 below is the right algebraic counterpart of the “velocity of deformation”. Indeed,  $X(F)_q$  associate with a point  $p \in M$  the velocity  $X(F)_q|_p \in T_{F(p,q)}N$  of the trajectory  $q \mapsto F(p, q)$  in  $N$ , where  $q$  is running along a trajectory of  $X$ .

**Definition 4.** *The  $F$ -relative vector field  $X(F)$  is the derivative of  $F$  with respect to  $X \in D(\mathcal{P})$ .*

In the case when  $\mathcal{P} \subset \mathbb{R}$  and  $X = \frac{d}{dt}$ , the derivative  $F' \stackrel{\text{def}}{=} \frac{d}{dt}(F)$  is called the *infinitesimal homotopy* associated with  $F$ , and symbols  $\frac{d}{dt}(F)_{t_0}$ ,  $\frac{dF}{dt}|_{t=t_0}$  and  $F'_{t_0}$  are interchangeable.

It is worth noticing that Proposition 1 does not hold if one consider, instead of  $\Gamma(\pi_M^\circ(\pi))$ , arbitrary families of sections along  $F$ , since the canonical lifting  $\tilde{X}$  needs *not* to be  $F$ -vertical. So, derivation(s) with respect to parameter(s) is not intrinsically defined in such a case. As explained in Remark 3, the reason is that sections of a family corresponding to different values of the parameter cannot be compared.

**Example 1** (Flow of a vector field). *Let  $X$  be a complete vector field on  $M$ ,  $\mathcal{P} = \mathbb{R}$ , and consider the time vector field on  $M \times \mathbb{R}$ , i.e., the  $\pi_M$ -vertical vector field*

$$(5) \quad \frac{\partial}{\partial t} \stackrel{\text{def}}{=} \tilde{\frac{d}{dt}}.$$

<sup>2</sup>When  $\mathcal{P} = [0, 1]$ ,  $F$  is an homotopy.

Then there exists a unique family of maps  $A : M \times \mathbb{R} \rightarrow M$ , such that  $A' = A^* \circ X$  and  $A_0 = \text{id}_M$ , i.e.,

$$(6) \quad \frac{\partial}{\partial t} \circ A^* = A^* \circ X \quad \text{and} \quad A \circ \iota_0 = \text{id}_M.$$

Family  $A$  fulfilling (6) is called the flow generated by  $X$ . Each member  $A_t$  is a diffeomorphism of  $M$ .

Observe that the infinitesimal homotopy  $A'$  determined by  $A$  can be interpreted as a family  $\{A'_t\}_{t \in \mathbb{R}}$  of vector fields, where  $A'_0 = X$ , but, in general,  $A'_t$  is the relative vector field  $A_t^* \circ X$ . This indicates the possibility that *any* homotopy may be seen as the “flow” associated with a family of *relative* vector fields  $\{X_t\}_{t \in \mathbb{R}}$ , and this analogy will lead, in a surprisingly straightforward way, directly to the proof of the Homotopy Formula (Section 9).

**Remark 5.** If  $A : M \times \mathcal{P} \rightarrow M$  is the flow of the vector field  $X \in D(M)$ , then

$$(7) \quad (M \times N) \times \mathcal{P} \ni (x, y, t) \xrightarrow{\tilde{A}} (A(x, t), y) \in M \times N$$

is the flow of the canonical lift  $\tilde{X}$ .

**Remark 6.** The canonical lift  $\tilde{X}$  of  $X \in D(M)$  (resp.,  $\tilde{Y}$  of  $Y \in D(\mathcal{P})$ ) is  $\iota_q$ -compatible with  $X$  (resp.,  $\iota_p$ -compatible with  $Y$ ) for all  $q \in \mathcal{P}$  (resp., for all  $p \in M$ ). Geometrically, the fact that  $\tilde{X}$  is  $\iota_q$ -compatible with  $X$ , means that, for any  $f \in C^\infty(M \times \mathcal{P})$  and  $p \in M$ , we have

$$(8) \quad \tilde{X}_{(p, q)}(f) = X_p(f_q), \quad q \in \mathcal{P},$$

i.e., the action of  $\tilde{X}$  on a family of functions  $f$  coincides with the action of  $X$  on each its member  $f_q$ .

#### 4. INTEGRATION OF FAMILIES

In this Section we will define the “inverse” of derivative operation, i.e., integration, by extending a bounded functional  $\mathcal{F} \in C^\infty(\mathcal{P})^\vee$  to families of objects, much as Corollary 1 allows to define the derivative operation by lifting derivations of  $\mathcal{P}$  to the cylinder. To this end, both  $C^\infty(\mathcal{P})$ , with  $\mathcal{P}$  compact, and  $C_c^\infty(\mathcal{P})$  are equipped with the norm of the maximum, so that  $\mathcal{F}$  is *bounded* if  $|\mathcal{F}(\varphi)| \leq K\|\varphi\|$  for a given  $K \geq 0$ , and all  $\varphi$ 's.

**Remark 7.** If  $\mathcal{F}$  is bounded and  $\varphi_t$  converges to  $\varphi$  point-wisely in  $C^\infty(\mathcal{P})$  as  $t \rightarrow 0$ , then  $\mathcal{F}(\varphi_t) \rightarrow \mathcal{F}(\varphi)$  in  $\mathbb{R}$  as  $t \rightarrow 0$ .

Let  $h_x \in \text{Spec}_\mathbb{R}(C^\infty(M))$  (see [4]) be the evaluation map at  $x \in M$ , and  $f \in C^\infty(M \times \mathcal{P})$ . Then, regarding  $f$  as a family of functions on  $\mathcal{P}$  parametrized by  $M$ , we can turn each its member  $f_x = \iota_x^*(f)$  into the real number  $\mathcal{F}(f_x)$ . In other words, family  $f$  is turned into the function  $\tilde{\mathcal{F}}(f) : x \mapsto \mathcal{F}(f_x)$ .

**Proposition 2.** Let  $\mathcal{F}$  be bounded,  $f \in C^\infty(M \times \mathcal{P})$  and  $\tilde{\mathcal{F}}(f)(x) \stackrel{\text{def}}{=} \mathcal{F}(f_x)$ . Then  $\tilde{\mathcal{F}}(f) \in C^\infty(M)$ ,

$$(9) \quad \begin{array}{ccc} C^\infty(M \times \mathcal{P}) & \xrightarrow{\tilde{\mathcal{F}}} & C^\infty(M) \\ \downarrow \iota_x^* & & \downarrow h_x \\ C^\infty(\mathcal{P}) & \xrightarrow{\mathcal{F}} & \mathbb{R} \end{array}$$

is a commutative diagram, and  $\tilde{\mathcal{F}}$  is  $C^\infty(M)$ -linear.

*Proof.* Obviously, if one replaces  $C^\infty(M)$  with  $\mathbb{R}^M$ , (9) becomes commutative, and

$$\tilde{\mathcal{F}}(hf) = \{\mathcal{F}(h(x)\iota_x^*(f))\}_{x \in M} = \{h(x)\mathcal{F}(\iota_x^*(f))\}_{x \in M} = h\tilde{\mathcal{F}}(f),$$

for  $h \in C^\infty(M)$  and  $f \in C^\infty(M \times \mathcal{P})$ , due to  $\mathbb{R}$ -linearity of  $\mathcal{F}$ , so  $C^\infty(M)$ -linearity holds too. It remains to be shown that  $\tilde{\mathcal{F}}(f) \in C^\infty(M)$ .

To this end, consider the flow  $\{A_t\}$  generated by a vector field  $X \in D(M)$ . Since  $(h_x \circ A_t^*)(\varphi) = A_t^*(\varphi)(x) = \varphi(A_t(x))$ ,  $\varphi \in \mathbb{R}^M$ , then  $h_x \circ A_t^* = h_{A_t(x)}$ . Therefore,

$$(10) \quad A_t^*(\tilde{\mathcal{F}}(f))(x) = (h_x \circ A_t^*)(\tilde{\mathcal{F}}(f)) = (h_{A_t(x)} \circ \tilde{\mathcal{F}})(f) = (\mathcal{F} \circ \iota_{A_t(x)}^*)(f).$$

Let  $\tilde{X} \in D(M \times \mathcal{P})$  be the canonical lift of  $X$  and  $\{\tilde{A}_t\}$  its flow (see Remark 5). Then  $\tilde{A}_t \circ \iota_x = \iota_{A_t(x)}$  and the existence of the derivative  $\frac{d}{dt}\big|_{t=0} \tilde{A}_t^*(f)$  implies the existence of the derivative

$$(11) \quad \frac{d}{dt}\bigg|_{t=0} (\mathcal{F} \circ \iota_x^*)(\tilde{A}_t^*(f))$$

(see Remark 7), which coincides with  $(\mathcal{F} \circ \iota_x^*)(\tilde{X}(f)) = (\mathcal{F}(\tilde{X}(f)))(x)$ . On the other hand, thanks to (10),  $(\mathcal{F} \circ \iota_x^*)(\tilde{A}_t^*(f)) = (\mathcal{F} \circ \iota_x^* \circ \tilde{A}_t^*)(f) = (\mathcal{F} \circ \iota_{A_t(x)}^*)(f) = A_t^*(\tilde{\mathcal{F}}(f))(x)$ . Define

$$(12) \quad X(\tilde{\mathcal{F}}(f))(x) \stackrel{\text{def}}{=} \frac{d}{dt}\bigg|_{t=0} A_t^*(\tilde{\mathcal{F}}(f))(x), \quad x \in M,$$

and observe that the derivative in (12) is well-defined because of (11). So, (12) reads

$$(13) \quad X(\tilde{\mathcal{F}}(f)) = \tilde{\mathcal{F}}(\tilde{X}(f)).$$

It follows immediately from (13) that the action of any differential operator  $\Delta = X_1 \circ \dots \circ X_s$ ,  $X_i \in D(M)$ , is well-defined on  $\tilde{\mathcal{F}}(f)$  and

$$(14) \quad \Delta(\tilde{\mathcal{F}}(f)) = \tilde{\mathcal{F}}(\tilde{X}_1(\dots \tilde{X}_s(f) \dots)).$$

This proves smoothness of  $\tilde{\mathcal{F}}(f)$ .<sup>3</sup> □

Since  $\tilde{\mathcal{F}}$  coincides with  $\mathcal{F}$  on the subalgebra  $C^\infty(\mathcal{P})$  of  $C^\infty(M \times \mathcal{P})$ , it is appropriate to call it the *lift* of  $\mathcal{F}$ .

**Proposition 3.** *Operator  $\tilde{\mathcal{F}}$  extends to a  $C^\infty(M)$ -homomorphism  $\tilde{\mathcal{F}}: \Gamma(\pi_M^\circ(\pi)) \rightarrow \Gamma(\pi)$ , denoted by the same symbol.*

*Proof.* Due to  $C^\infty(M)$ -linearity of  $\mathcal{F}$ , the the product  $\tilde{\mathcal{F}} \otimes \text{id}$  is a well-defined  $C^\infty(M)$ -homomorphism from  $C^\infty(M \times \mathcal{P}) \otimes_{C^\infty(M)} \Gamma(\pi)$  to  $\Gamma(\pi)$ . □

If  $\mathcal{P}$  is compact and  $\mathcal{F} = \int_{\mathcal{P}}$ , then the section

$$\widetilde{\int_{\mathcal{P}}} \sigma \in \Gamma(\pi),$$

is called the *integral* (over  $\mathcal{P}$ ) of the family of sections  $\sigma \in \Gamma(\pi_M^\circ(\pi))$ . Corollary 2 below is a straightforward generalization of an elementary property of differentiation, namely that it commutes with integration w.r.t. other parameter(s).

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<sup>3</sup>If a subset  $A \subseteq \mathbb{R}^M$  is closed under the action of differential operators on  $M$ , then  $A \subseteq C^\infty(M)$ .



**Corollary 2.** *Let  $X : C^\infty(M) \longrightarrow \Gamma(\pi)$  be a derivation and  $\tilde{X}$  its canonical lift. Then the diagram*

$$(15) \quad \begin{array}{ccc} C^\infty(M) & \xrightarrow{X} & \Gamma(\pi) \\ \tilde{\mathcal{F}} \uparrow & & \uparrow \tilde{\mathcal{F}} \\ C^\infty(M \times \mathcal{P}) & \xrightarrow{\tilde{X}} & \Gamma(\pi_M^\circ(\pi)), \end{array}$$

*is commutative.*

*Proof.* Locally,  $X = X_i \otimes s^i$ , where  $X_i \in D(M)$  and  $\{s^i\} \subseteq \Gamma(\pi)$  is a local basis. Then  $\tilde{X} = \tilde{X}_i \otimes s^i$  and  $\tilde{X}(f) = \tilde{X}_i(f) \otimes s^i$ , for every  $f \in C^\infty(M \times \mathcal{P})$ , so that  $\tilde{\mathcal{F}}(\tilde{X}(f)) = \tilde{\mathcal{F}}(\tilde{X}_i(f)) \otimes s^i$  (see Proposition 3). But, in view of (13),  $\tilde{\mathcal{F}}(\tilde{X}_i(f)) \otimes s^i$  coincides with  $X_i(\tilde{\mathcal{F}}(f)) \otimes s^i$ , i.e.,  $X(\tilde{\mathcal{F}}(f))$ .  $\square$

Proposition 4 below provides a sort of Newton–Leibniz formula depending on parameters running over  $M$ .

**Proposition 4.** *If  $F : M \times [a, b] \longrightarrow N$ , then*

$$(16) \quad \int_a^b \circ F' = F_b^* - F_a^*.$$

*Proof.* By evaluating both sides of (16) on  $f \in C^\infty(N)$  and then on  $x \in M$  we obtain

$$(17) \quad \left( h_x \circ \int_a^b \circ F' \right) (f) = f(F_b(x)) - f(F_a(x)).$$

In view of Definition 4 and of commutativity of (9), the left-hand side of (17) reads

$$(18) \quad \left( \int_a^b \circ \iota_x^* \circ \frac{\partial}{\partial t} \right) (F^*(f)).$$

In its turn (see Remark 6), (18) reads  $\left( \int_a^b \circ \frac{d}{dt} \right) (\iota_x^*(F^*(f)))$ , i.e.,  $(\iota_x^*(F^*(f)))(b) - (\iota_x^*(F^*(f)))(a)$ , which is precisely the right-hand side of (17).  $\square$

An interesting case of (16) is obtained when  $N = M \times \mathbb{I}$  and  $F = \text{id}_{M \times \mathbb{I}}$ . Indeed,  $F_t = \iota_t$ , and (16) becomes

$$(19) \quad \iota_b^* - \iota_a^* = \int_a^b \circ \frac{\partial}{\partial t}.$$

Formula (19) is more general than (16), in that the family to be integrated does not need to belong to  $\text{im } F^*$ . Both (16) and (19) admits a generalization to families of forms (see Section 8), which is essential to prove the homotopy formula (see Section 9). It should be stressed that an analogous generalization to sections of a generic vector bundle  $\pi$  along a map  $F : M \times \mathbb{I} \rightarrow N$  is not always possible. Indeed, for a generic  $\pi$ , the “derivative”  $F'$  of  $F$  is not defined, unless  $\pi$  defines a “covariant quantity”, as explained in Section 5 below.

## 5. FAMILIES OF COVARIANT QUANTITIES

Roughly speaking, a *covariant quantity* on  $M$  is a bundle  $\pi_{\Phi, M}$  which is *naturally associated* with  $M$ . Naturality implies, in particular, the existence of pull-backs and Lie derivatives of sections of  $\pi_{\Phi, M}$ . The formers allow to define families of covariant quantities, and the latter to formalize their derivative.

More precisely, let  $\mathbf{F}$  be a representable functor of differential calculus over  $M$ , and  $\Phi(M)$  the  $C^\infty(M)$ -module which represents it in the category of geometric  $C^\infty(M)$ -modules. Observe that a vector bundle  $\pi_{\Phi,M}$  exists, such that  $\Phi(M) = \Gamma(\pi_{\Phi,M})$ .

**Definition 5.**  $\Theta \in \Phi(M)$  is a covariant quantity (of type  $\Phi$ ) on  $M$ .

Since  $M \mapsto \Phi(M)$  is a functor, any  $F : M \rightarrow N$  determines the functorial pull-back  $F^* : \Phi(N) \rightarrow \Phi(M)$ . On the other hand, being  $\Phi(N)$  a module of sections, there is also a bundle-theoretic pull-back  $F^\circ : \Gamma(\pi_{\Phi,N}) \rightarrow \Gamma(F^\circ(\pi_{\Phi,N}))$ . In general, however, the module  $\Gamma(F^\circ(\pi_{\Phi,N}))$  has nothing in common with  $\Gamma(\pi_{\Phi,M}) = \Phi(M)$ , being neither a submodule of  $\Gamma(\pi_{\Phi,M})$  nor an its quotient.

**Definition 6.** Elements of the submodule  $\Phi_F(M) \subseteq \Phi(M)$  generated by  $F^*(\Phi(N))$  are called  $F$ -horizontal.

In other words,  $\Phi_F(M)$  is made of elements of  $\Phi(M)$  which are of the form  $f_i F^*(\omega^i)$ , with  $f_i \in C^\infty(M)$ ,  $\omega^i \in \Phi(N)$ . In particular,  $\Phi_{\pi_M}(M \times \mathcal{P})$  is precisely the module of families of sections of  $\pi_{\Phi,M}$  according to Definition 2. Indeed, when  $F = \pi_M$ , one has  $F^\circ = F^*$  and  $\Gamma(\pi_M^\circ(\pi_{\Phi,M})) = \Phi_{\pi_M}(M \times \mathcal{P})$ . So, for covariant quantities, the vague idea of being sliceable is properly formalized in terms of  $\pi_M$ -horizontal.

**Definition 7.** Elements of  $\Phi_{\pi_M}(M \times \mathcal{P})$  are families of  $\Phi$ -type quantities on  $M$ .

**Example 2.** The module  $\Lambda^k(M)$  represents the functor  $D_k : P \rightarrow D_k(P)$  (see [4]). By definition,  $\mathbf{F} = D_0$  is the identity functor and  $\Phi(M) = \Lambda^0(M) = C^\infty(M)$ . The Newton-Leibniz formula (16) for families of maps corresponds, therefore, to  $\Phi = \Lambda^0$ , and it will be generalized to  $\Lambda^k$ ,  $k > 0$ , in Section 9.

Definition 7 allows to differentiate families of  $\Phi$ -type quantities on  $M$  by means of Lie derivatives. More precisely, given a vector field  $X \in D(M)$  and its flow  $A$  (see Example 1), the Lie derivative  $L_X^\Phi : \Phi(M) \rightarrow \Phi(M)$  is the derivative

$$(20) \quad L_X^\Phi \stackrel{\text{def}}{=} \left. \frac{d A_{t,\Phi}^*}{dt} \right|_{t=0}$$

of the family of functorial pull-backs  $A_{t,\Phi}^* : \Phi(N) \rightarrow \Phi(N)$ . When  $\Phi = \Lambda$ , the Cartan formula allows to define  $L_X^\Phi$  in a pure algebraic way,<sup>4</sup> namely

$$L_X^\Lambda \stackrel{\text{def}}{=} i_X \circ d + d \circ i_X.$$

So, the Lie derivative  $L_X^\Phi$  act as well on the submodule  $\Phi_{\pi_M}(M \times \mathcal{P}) \subseteq \Phi(M)$  and on the subspace  $\pi_M^*(\Phi(M)) \subseteq \Phi_{\pi_M}(M \times \mathcal{P})$ . Corollary 3 below shows that  $L_X^\Phi$  actually vanishes on  $\pi_M^*(\Phi(M))$ , while on  $\Phi_{\pi_M}(M \times \mathcal{P}) \subseteq \Phi(M)$  it coincides with the derivative operator  $\nabla_X$  defined in Proposition 1.

**Proposition 5.**  $L_X^\Phi$  preserves  $\Gamma(\pi_M^\circ(\xi))$  and is  $\pi_M$ -vertical.

*Proof.* Let  $\tilde{A}_t$  be the flow generated by  $\tilde{X}$ . Then  $\pi_M \circ \tilde{A}_t = \pi_M$  (see Corollary 1) and, in view of (20),

$$L_X^\Phi \circ \pi_M^* = \left. \frac{d \tilde{A}_{t,\Phi}^*}{dt} \right|_{t=0} \circ \pi_M^* = \left. \frac{d \tilde{A}_{t,\Phi}^* \circ \pi_M^*}{dt} \right|_{t=0} = \left. \frac{\pi_M^*}{dt} \right|_{t=0} = 0.$$

So,  $L_X^\Phi$  is  $\pi_M$ -vertical. Let now  $\rho = f_i \pi_M^*(\Theta^i) \in \Gamma(\pi_M^\circ(\xi))$ , with  $f_i \in C^\infty(M \times \mathcal{P})$  and  $\Theta^i \in \Phi(M)$ . Then  $L_X^\Phi(\rho) = X(f_i) \pi_M^*(\Theta^i) \in \Gamma(\pi_M^\circ(\xi))$  follows from Leibniz rule and  $\pi_M$ -verticality of  $L_X^\Phi$ .  $\square$

<sup>4</sup>The algebraic definition of the Lie derivative for arbitrary  $\Phi$ 's is a more delicate problem, which do not touch here.

Proposition 5 allows an immediate proof of Corollary 3 below.

**Corollary 3.**

- (1)  $L_X^\Phi$  restricted to  $\Gamma(\pi_M^\circ(\xi))$  coincides with  $\nabla_X$ ,
- (2)  $\pi_M^*((\Phi(M)))$  consists of constant families of  $\Phi$ -type covariant quantities.

In the case of  $\Phi = \Lambda$ , Definition 7 corresponds to the well-known notion of horizontal forms, i.e., families of forms according to Definition 6. Details will be discussed in Section 8. Section 6 below focuses on the dual side, i.e., families of vector fields.

6. EXAMPLE OF FAMILIES OF “CONTROVARIANT QUANTITIES”: VECTOR FIELDS AND DERIVATIONS

Following paradigm (2), a family  $\{Z_g\}$  of vector fields should correspond to an element  $Z \in D(M \times \mathcal{P})$ . But  $Z$  is a contravariant quantity, so the pull-back  $\iota_q^*$  cannot be applied to it. On the other hand,  $Z$  may be interpreted as a section of the tangent bundle (see Remark 2), and as such the bundle-theoretic pull-back can be applied to it. However, the so-obtained vector field  $\iota_q^\circ(Z)$  is a *relative* one.

In this Section we implement correspondence (3) in the case of vector fields. The first step is to put

$$(21) \quad Z_q \stackrel{\text{def}}{=} \iota_q^* \circ Z \circ \pi_M^*.$$

The reason of choice (21) is that  $Z_q$  is precisely the restriction  $Z|_{M \times \{q\}}$  via identification  $\iota_q$  (see Remark 1). The second task is to determine which submodule should be replaced to  $D(M \times \mathcal{P})$  in order to make

$$(22) \quad D(M \times \mathcal{P}) \ni Z \longmapsto \{Z_g\}_{g \in \mathcal{P}}$$

a bijection, i.e., to discover what is the right formalization of a *sliceable vector field*. Following intuition, a vector field  $Z$  is *sliceable* if it is tangent to all the slices  $M_q$ 's, i.e., if it is  $\pi_{\mathcal{P}}$ -vertical. This motivates Definition 8 below.

**Definition 8.** A  $\mathcal{P}$ -parametrized family of vector fields on a manifold  $M$  is a  $\pi_{\mathcal{P}}$ -vertical field on  $M \times \mathcal{P}$ .

So, unlike a family of covariant quantities on  $M$ , which is a  $\pi_M$ -horizontal quantity on  $M \times \mathcal{P}$ , a family of such “contravariant quantities” as vector fields on  $M$ , is made of  $\pi_{\mathcal{P}}$ -vertical quantities on  $M \times \mathcal{P}$ . Nonetheless, Proposition 6 below shows that Definition 8 above—much as Definition 7 for covariant quantities—is but a particular cases of Definition 2.

To this end, notice that a  $\pi_{\mathcal{P}}$ - (resp.,  $\pi_M$ -)vertical vector field  $Z$  on the cylinder  $M \times \mathcal{P}$  is uniquely determined by its restriction  $Z|_{C^\infty(M)}$  (resp.,  $Z|_{C^\infty(\mathcal{P})}$ ) since, by definition,  $Z$  vanishes on  $C^\infty(\mathcal{P})$  (resp.,  $C^\infty(M)$ ) (see Lemma 1). But  $Z|_{C^\infty(M)}$  is a  $C^\infty(M \times \mathcal{P})$ -valued derivation of  $C^\infty(M)$  (resp.,  $C^\infty(\mathcal{P})$ ), i.e., a relative vector field along the map  $\pi_M$  (resp.,  $\pi_{\mathcal{P}}$ ).

**Remark 8.** In the same coordinates as Remark 4, it is easy to see that  $Z$  is a vector field along  $\pi_M$  if and only if  $Z = Z^i \frac{\partial}{\partial x^i}$ ,  $Z^i \in C^\infty(M \times \mathcal{P})$ .

On the other hand, a vector field  $Z$  along  $\pi_M$  is a section of the induced bundle  $\pi_M^\circ(\tau_M)$  (see [4]). Moreover,

$$(23) \quad D_{\pi_M}(M \times N) \stackrel{\text{def}}{=} \Gamma(\pi_M^\circ(\tau_M))$$

is a sub- $C^\infty(M \times \mathcal{P})$  of  $D(M \times \mathcal{P})$ , naturally isomorphic to  $C^\infty(M \times N) \otimes_{C^\infty(M)} D(M)$ . Similarly if  $Z$  is a vector field along  $\pi_{\mathcal{P}}$ . In other words, it is natural to identify  $\pi_{\mathcal{P}}$ - (resp.,  $\pi_M$ -)vertical vector fields with vector fields along  $\pi_M$  (resp.,

$\pi_{\mathcal{P}}$ ). Proposition 6 below, whose easy proof is omitted, shows the functoriality of such identification.

**Remark 9.** For any  $C^\infty(M \times \mathcal{P})$ -module  $P$ , define the submodule  $D_{\pi_M}(P) \subseteq D(P)$  of  $P$ -valued derivations of  $C^\infty(M \times \mathcal{P})$  along  $\pi_M$ . Then the correspondence  $P \mapsto D_{\pi_M}(P)$  is a functor.

**Proposition 6.** Functor  $D_{\pi_M}$  (resp.,  $D_{\pi_{\mathcal{P}}}$ ) is naturally identified with functor of  $\pi_{\mathcal{P}}$ - (resp.,  $\pi_M$ -) vertical derivations.

Now that Definition 8 has become a particular case of Definition 2, it can be generalized to arbitrary differential operators.

**Definition 9.** A  $\mathcal{P}$ -parametrized family of differential operators between  $C^\infty(M)$ -modules  $P$  and  $Q$  is a  $C^\infty(\mathcal{P})$ -linear differential operator

$$(24) \quad \Delta : C^\infty(M \times \mathcal{P}) \otimes_{C^\infty(M)} P \mapsto C^\infty(M \times \mathcal{P}) \otimes_{C^\infty(M)} Q.$$

If  $P = \Gamma(\eta)$  and  $Q = \Gamma(\xi)$ , then operator  $\Delta$  from Definition 9 can be naively interpreted as a family  $\{\Delta_q\}_{q \in \mathcal{P}}$  of  $\xi$ -valued differential operator on  $\eta$ . Indeed, in this case (24) reads

$$\Delta : \sigma \in \Gamma(\pi_M^\circ(\eta)) \mapsto \Delta(\sigma) \in \Gamma(\pi_M^\circ(\xi)),$$

i.e.,  $\Delta$  maps a family  $\sigma$  of sections of  $\eta$  into a family  $\Delta(\sigma)$  of section of  $\xi$ , in such a way that  $\Delta(\sigma)_q = \Delta_q(\sigma_q)$ , where  $\Delta_q \in \text{Diff}(\Gamma(\eta), \Gamma(\xi))$  (see Definition 2).

**Example 3.** A  $\mathcal{P}$ -parametrized family of derivations of the algebra  $C^\infty(M)$  with values in a  $C^\infty(M)$ -module  $P$  is a  $C^\infty(\mathcal{P})$ -linear derivation  $Z : C^\infty(M \times \mathcal{P}) \rightarrow C^\infty(M \times \mathcal{P}) \otimes_{C^\infty(M)} P$ . By using the same coordinates as Remark 4, and a local basis  $\{s^j\}$  of  $P = \Gamma(\xi)$ , a family  $Z$  of  $P$ -valued derivations can be represented as  $Z = Z_j^i \otimes \frac{\partial}{\partial x^i} \otimes s^j$ , with  $Z_j^i \in C^\infty(M \times \mathcal{P})$ . Accordingly,  $Z_q = \iota_q^*(Z_j^i) \frac{\partial}{\partial x^i} \otimes s^j$ .

## 7. VERTICAL AND HORIZONTAL DIFFERENTIAL FORMS

Proposition 6 above identifies the notion of a *sliceable derivation* and, in particular, of a family of vector fields, with the functor  $D_{\pi_M}$  of derivations along the canonical projection  $\pi_M$ . In this Section we show that  $D_{\pi_M}$  is representable, and that its representative object is precisely the module of *horizontal* differential 1-forms, which, in its turn, according to Definition 7, corresponds to families of differential 1-forms.<sup>5</sup> In other words, families of differential forms can be thought of as the representative object of families of vector fields, in total agreement with the logic of differential calculus.

Since families of vector fields correspond not only to the vector fields along  $\pi_M$ , but also to the vertical  $\pi_{\mathcal{P}}$ -vector fields (Proposition 6), it is natural to look for the representative object of vertical derivations. As Lemma 3 below shows, such an object is the quotient of  $\Lambda^1(M)$  w.r.t. the submodule of horizontal forms, in the sense of Definition 6. However, when  $\Phi = \Lambda^1$ , Definition 6 gains an important geometrical meaning, so it is worth specializing it here.

In order to have the most general definition, let  $f : M \rightarrow N$  be a smooth map.

**Definition 10.** The sub-module  $\Lambda_f^1(M)$  of  $\Lambda^1(M)$  generated by the image of  $f^*$  is the module of  $(f-)$ horizontal 1-forms on  $M$ .

Geometrically, a 1-form  $\omega$  is *horizontal* when it is constant along the fibers of  $f$ , i.e.,  $i_X(\omega) = 0$  for all  $f$ -vertical vector fields  $X \in D_f^v(M)$ . Obviously,  $i_X(f^*(\eta)) = 0$ , i.e., a 1-form which is horizontal in the sense of Definition 10 is also horizontal in the geometrical sense. Lemma 3 below shows that the converse holds as well.

<sup>5</sup>Indeed, differential 1-forms are special type of covariant quantities (see Section 5).

Put

$$(25) \quad \bar{\Lambda}_f^1(M) \stackrel{\text{def}}{=} \frac{\Lambda^1(M)}{\Lambda_f^1(M)}.$$

**Lemma 3.**

$$D_f^v(M) \cong \text{Hom}(\bar{\Lambda}_f^1(M), C^\infty(M)).$$

*Proof.* In view of the isomorphism

$$(26) \quad D(M) \ni X \leftrightarrow i_X \in \text{Hom}(\Lambda^1(M), C^\infty(M)),$$

a submodule  $Q \subseteq \Lambda^1(M)$  must exist, such that  $D_f^v = \text{Ann}(Q)$ . But a vector field  $X \in D(M)$  is  $f$ -vertical if and only if  $i_X(\omega) = 0$  for any  $\omega \in \Lambda_f^1(M)$ , so  $Q = \Lambda_f^1(M)$ .  $\square$

**Definition 11.** An element  $[\omega]_{\Lambda_f^1(M)}$  of the quotient module (25) is called an ( $f$ -)vertical 1-form, and is denoted by  $\bar{\omega}$ .

**Corollary 4.** The functor  $D_f^v$  is represented by the module of  $f$ -vertical 1-forms, and the natural embedding  $D_f^v \subseteq D$  corresponds to the canonical projection  $\Lambda^1(M) \longrightarrow \frac{\Lambda^1(M)}{\Lambda_f^1(M)}$  of representative objects.

Consider now the cylinder  $M \times \mathcal{P}$ . In this case, the peculiar geometry of the manifold  $M \times \mathcal{P}$  allows to identify vertical forms with respect to one projection with horizontal forms with respect to the other one. Details are as follows.

**Lemma 4.**

$$(27) \quad \Lambda^1(M \times \mathcal{P}) = (\Lambda^1(M) \bar{\otimes}_{\mathbb{R}} C^\infty(\mathcal{P})) \oplus (C^\infty(M) \bar{\otimes}_{\mathbb{R}} \Lambda^1(\mathcal{P})),$$

*Proof.* See [2].  $\square$

**Proposition 7.**  $\pi_M$ - (resp.,  $\pi_{\mathcal{P}}$ -)vertical 1-forms on  $M \times \mathcal{P}$  are identified with  $\pi_{\mathcal{P}}$ - (resp.,  $\pi_M$ -)horizontal 1-forms.

*Proof.* To prove both assertions, it suffices to interpret (27) as

$$(28) \quad \Lambda^1(M \times \mathcal{P}) = \Lambda_{\pi_M}^1(M \times \mathcal{P}) \oplus \Lambda_{\pi_{\mathcal{P}}}^1(M \times \mathcal{P}).$$

In fact, (28) follows from (27) since  $\Lambda^1(M) \bar{\otimes}_{\mathbb{R}} C^\infty(\mathcal{P})$  coincides with  $\Lambda^1(M) \otimes_{C^\infty(M)} C^\infty(M \times \mathcal{P})$ ,<sup>6</sup> which is the submodule of  $\Lambda^1(M \times \mathcal{P})$  generated by  $\text{im } \pi_M^*$ . The same for  $\Lambda_{\pi_{\mathcal{P}}}^1(M \times \mathcal{P})$ .  $\square$

Combining Proposition 7 above with Proposition 6 and Corollary 4, we easily obtain the next Corollary 5.

**Corollary 5.** Functor  $D_{\pi_M}$  (resp.,  $D_{\pi_{\mathcal{P}}}$ ) is represented by  $\Lambda_{\pi_M}^1(M \times \mathcal{P})$  (resp.  $\Lambda_{\pi_{\mathcal{P}}}^1(M \times \mathcal{P})$ ) in the category of geometric  $C^\infty(M \times \mathcal{P})$ -modules.

In other words, derivations along  $\pi_M$  (resp.,  $\pi_{\mathcal{P}}$ ) may be called  $\pi_M$ - (resp.,  $\pi_{\mathcal{P}}$ -)horizontal derivations, since they are represented by  $\pi_M$ - (resp.,  $\pi_{\mathcal{P}}$ -)horizontal differential 1-forms. This way, in total analogy with covariant quantities, sliceable derivations coincide with the horizontal ones, and Corollary 5 reads as the “horizontal version” of the duality (26) between derivations and 1-forms.

In order to define families of higher-order differential forms, it is convenient to denote by  $\Lambda^{p,q}(M \times \mathcal{P})$  the submodule of  $\Lambda^{p+q}(M \times \mathcal{P})$  generated by  $\pi_M^*(\Lambda^p(M)) \otimes_{C^\infty(M \times \mathcal{P})} \pi_{\mathcal{P}}^*(\Lambda^q(\mathcal{P}))$ .

**Definition 12.** Elements of  $\Lambda^{p,q}(M \times \mathcal{P})$  are the forms of type  $(p, q)$  on the cylinder  $M \times \mathcal{P}$ .

<sup>6</sup>See [2] concerning the smoothened tensor product.

**Corollary 6.** *The direct sum decomposition*

$$(29) \quad \Lambda^k(M \times \mathcal{P}) = \bigoplus_{i=0}^k \Lambda^{i,k-i}(M \times \mathcal{P})$$

*holds.*

*Proof.* Easy follows from (28).  $\square$

Definition 13 below complies with the general definition of families of objects (see Definition 2).

**Definition 13.** *A  $\mathcal{P}$ -parametrized family of  $k$ -forms is an element of the module  $\Lambda_{\pi_M}^k(M \times \mathcal{P}) = \Lambda^{k,0}(M \times \mathcal{P})$ .*

However, the geometrical content of Definition 13 is not self-evident, and we may look for alternative ways to define families of higher order differential forms. Geometrically, since horizontal 1-forms are annihilated by vertical vector fields (see Lemma 3), horizontal  $k$ -forms may be defined as those that are annihilated by vertical  $k$ -multivector fields, namely,

$$(30) \quad \mathcal{I}_f \stackrel{\text{def}}{=} \{\omega \in \Lambda^+(M) \mid \omega(X_1, \dots, X_k) = 0 \quad \forall X_i, \dots, X_k \in D^v, k = \deg(\omega)\}.$$

But it is also possible to generalize Definition 10, so that horizontal forms of positive degree are given by the ideal  $\langle f^*(\Lambda^+(N)) \rangle$  of  $\Lambda(M)$  generated by the image via  $f^*$  of positive degree forms  $\Lambda^+(N)$  on  $N$ .

Obviously,  $\langle f^*(\Lambda^+(N)) \rangle \subseteq \mathcal{I}_f$  and  $\mathcal{I}_f$  is a differential ideal of  $\Lambda(M)$ .

**Lemma 5.** *If  $f$  is regular, then  $\mathcal{I}_f = \langle f^*(\Lambda^+(N)) \rangle$ .*

*Proof.* Well-known result in the theory of distributions.  $\square$

So, under regularity conditions for  $f$  (always assumed in the sequel), the algebraic notion of horizontal forms is geometrically interpreted in the context of distributions, thus motivating Definition 14 below.

**Definition 14.**  *$\langle f^*(\Lambda^+(N)) \rangle$  is the ideal of horizontal ( $f$ )-forms.*

To unveil the geometrical content of Definition 14, observe that the family

$$(31) \quad M \ni x \longmapsto \ker d_x f \subseteq T_x M$$

of tangent subspaces is a Fröbenius distribution, whose maximal integral submanifolds are the fibers of  $f$ . (31) is called the  *$f$ -vertical distribution on  $M$*  and is denoted by  $\mathcal{V}_f$ . Then the annihilator  $\mathcal{V}_f \Lambda(M)$  of  $\mathcal{V}_f$  is the ideal  $\mathcal{I}_f$  defined by (30), i.e., the ideal of horizontal  $f$ -forms. Observe that  $D_f^v(M)$  is precisely the module of vector fields belonging to  $\mathcal{V}_f$ , and  $D_f^v(M)_x = \ker d_x f$ .

**Lemma 6.** *For any  $x \in M$ , the space of skew-symmetric  $k$ -multilinear forms on  $D_f^v(M)_x$  identifies with  $(\mathcal{I}_f \cap \Lambda^k(M))_x$ .*

*Proof.* First recall that (see [3])

$$(32) \quad \text{Ann}(L^{\wedge k}) = \{\varphi \in (V^{\wedge k})^\vee \mid \varphi|_{L^{\wedge k}} = 0\} = (L^{\wedge k})^\vee.$$

If  $L = \ker d_x f$  and  $V = T_x M$ , then  $(V^{\wedge k})^\vee$  identifies with  $(T_x^*(M))^k$  and  $\text{Ann}(L^{\wedge k})$  with  $(\mathcal{I}_f \cap \Lambda^k(M))_x$ . To conclude the proof, observe that  $(L^{\wedge k})^\vee$  is the space of skew-symmetric  $k$ -multilinear forms on  $D_f^v(M)_x$ .  $\square$

Lemma 6 motivates Definition 15 below.

**Definition 15.**  $f$ -vertical differential forms, usually denoted by  $\bar{\omega}$ , are elements of

$$(33) \quad \bar{\Lambda}_f(M) = \frac{\Lambda(M)}{\mathcal{I}_f}.$$

Indeed,  $\bar{\Lambda}_f^k(M)$  can be interpreted as the “co-distribution” on  $M$

$$(34) \quad M \ni x \mapsto \bar{\Lambda}_f^k(M)_x = \frac{(T_x^* M)^{\wedge k}}{\text{Ann}((\ker d_x f)^{\wedge k})} = (D_f^v(M)_x^{\wedge k})^\vee,$$

dual to the  $k$ -th power of the vertical distribution (31). Corollary 7 below is the “global analog” of Lemma 6.

**Corollary 7.** *An element  $\bar{\omega} \in \bar{\Lambda}_f(M)$ ,  $\omega \in \Lambda^k(M)$ , is naturally interpreted as the  $k$ -multilinear skew-symmetric  $C^\infty(M)$ -multilinear map*

$$(35) \quad \underbrace{D^v(M) \times \cdots \times D^v(M)}_{k \text{ times}} \ni (X_1, \dots, X_k) \mapsto \omega(X_1, \dots, X_k) \in C^\infty(M).$$

Definition 15 makes it clear that  $\bar{\Lambda}_f(M)$  is not only the dual to the module of vertical  $k$ -multivectors, but it also inherits the quotient differential algebra structure from  $\Lambda(M)$ . In its turn, such a differential algebra structure will be used in Section 8 below to define a differential algebra structure on families of forms.

## 8. FAMILIES OF DIFFERENTIAL FORMS AND THEIR NATURAL OPERATIONS

Among all geometrical quantities, differential forms are perhaps the most interesting one, due to the rich structure they possess. This is reflected by the variety of equivalent ways in which families of differential forms can be defined. Namely, a  $\mathcal{P}$ -parametrized family of differential forms is

- an element of  $\Gamma(\pi_M^\circ(\xi))$ , where  $\xi = \bigoplus_k (\tau_M^*)^{\wedge k}$  (Definition 2),
- a  $\pi_M$ -horizontal  $\Lambda$ -type covariant quantity (Definition 7),
- an element of the direct sum  $\bigoplus_k \Lambda^{k,0}(M \times \mathcal{P})$  (Definition 13).

Independently on the definition, the symbol  $\Lambda_{\pi_M}(M \times \mathcal{P})$  will be used for the module of families of differential forms. It should be noticed that none of the definitions above shows that  $\Lambda_{\pi_M}(M)$  possesses a differential algebra structure—so yet another perspective is needed.

To this end, it is enough to notice that the kernel of the correspondence (2) which turns *any* form into a family coincides with the ideal  $\mathcal{I}_{\pi_{\mathcal{P}}}$  of  $\pi_{\mathcal{P}}$ -horizontal forms. In other words, a  $\mathcal{P}$ -parametrized family of differential forms is also

- an element of the *differential algebra*  $\bar{\Lambda}_{\pi_{\mathcal{P}}}(M \times \mathcal{P})$  of  $\pi_{\mathcal{P}}$ -vertical differential forms (see Definition 15).<sup>7</sup>

The drawback of the last definition is that, unlike the first three ones, it does not make  $\Lambda_{\pi_M}(M \times \mathcal{P})$  a submodule of  $\Lambda(M \times \mathcal{P})$ , but rather an its quotient. Decomposition (29) allows to treat the last definition on the same footing as the others, thus obtaining a submodule of  $\Lambda(M \times \mathcal{P})$  which is also a differential algebra. More precisely, introduce the canonical projections  $p_{i,k-i} : \Lambda^k(M \times \mathcal{P}) \rightarrow \Lambda^{i,k-1}(M \times \mathcal{P})$ .

**Definition 16.** *The  $M$ -horizontalization of  $k$ -forms is  $p_{k,0} : \omega \mapsto \bar{\omega} \stackrel{\text{def}}{=} p_{k,0}(\omega)$ , while the  $M$ -horizontalization is  $p_0 \stackrel{\text{def}}{=} \bigoplus p_{k,0}$ .*

<sup>7</sup>The same correspondence between families of quantities and vertical quantities was found in the context of vector fields (see Proposition 6).

Put  $\bar{d} \stackrel{\text{def}}{=} p_{k,0} \circ d|_{\Lambda_{\pi_M}^k(M \times \mathcal{P})} : \Lambda_{\pi_M}^k(M \times \mathcal{P}) \longrightarrow \Lambda_{\pi_M}^{k+1}(M \times \mathcal{P})$ . Then, for any  $\omega \in \Lambda_{\pi_M}^k(M \times \mathcal{P})$ ,  $d^2\omega$  is exactly the  $(k+2,0)$ -component of the form  $d^2\omega$ . So,  $\bar{d}^2 = 0$ . Notice also that the differential  $\bar{d}$  on  $\Lambda_{\pi_M}(M \times \mathcal{P})$  is precisely the one induced from the differential of  $\bar{\Lambda}_{\pi_{\mathcal{P}}}(M \times \mathcal{P})$  via the identification

$$\bar{\Lambda}_{\pi_{\mathcal{P}}}(M \times \mathcal{P}) \cong \Lambda_{\pi_M}(M \times \mathcal{P}),$$

due to decomposition (29). So, a family of differential forms is also

- an element of the differential algebra  $(\Lambda_{\pi_M}(M \times \mathcal{P}), \bar{d})$ .

The last point of view is the most complete and useful one, thus motivating Definition 17 below.

**Definition 17.**  $(\Lambda_{\pi_M}(M \times \mathcal{P}), \bar{d})$  is the  $\pi_M$ -horizontal de Rham complex, and  $\bar{d}$  is the horizontal differential.

Definition 17 allows to formalize the well-known interchangeability of derivative and integration in the context of families of forms. Indeed, integration of families (see Section 4) becomes more interesting when it is performed on differential forms, since it interacts with the horizontal differential. More precisely, as Corollary 8 below shows, if  $\mathcal{P}$  to be compact and  $\mathcal{F} \in C^\infty(\mathcal{P})^\vee$  is bounded, then the lift  $\tilde{\mathcal{F}}$  is a  $\Lambda(M)$ -linear cochain map.

To this end, regard the 1-st de Rham differential  $d : C^\infty(M) \longrightarrow \Lambda^1(M)$  as a  $\Lambda_{\pi_M}^1(M)$ -valued map, and construct its  $\pi_{\mathcal{P}}$ -vertical lift  $\tilde{d}$  as in Proposition 1.

**Proposition 8.**  $\tilde{d}$  coincides with  $\bar{d}$  and the diagram

$$(36) \quad \begin{array}{ccc} C^\infty(M) & \xrightarrow{d} & \Lambda^1(M) \\ \tilde{\mathcal{F}} \uparrow & & \uparrow \tilde{\mathcal{F}} \\ C^\infty(M \times \mathcal{P}) & \xrightarrow{\tilde{d}} & \Lambda_{\pi_M}^1(M \times \mathcal{P}) \end{array}$$

is commutative.

*Proof.* Enough to show that  $\bar{d}$  fulfills the properties of the canonical lift of  $d$  (see Proposition 1). Indeed, if  $f \in C^\infty(M)$ , then  $\bar{d}(\pi_M^*(f)) = \bar{d}\pi_M^*(f) = \pi_M^*(df)$ . So  $\bar{d}$  extends  $d$ . Moreover, if  $g \in C^\infty(\mathcal{P})$ , then  $\bar{d}(\pi_{\mathcal{P}}^*(g)) = 0$ , since  $\pi_{\mathcal{P}}^*(dg) \in \Lambda^{0,1}(M \times \mathcal{P})$ . So,  $\bar{d}$  is  $\pi_{\mathcal{P}}$ -vertical.  $\square$

**Corollary 8.** Operator  $\tilde{\mathcal{F}}$  is a cochain map from  $\Lambda_{\pi_M}(M \times \mathcal{P})$  to  $\Lambda(M)$ , of degree 0 and  $\Lambda(M)$ -linear.

*Proof.* A horizontal  $k$ -form  $\omega$  can be presented as  $\omega = f_i \pi_M^*(\eta^i)$ , with  $\eta_i \in \Lambda^k(M)$ , and it is identified with  $f_i \otimes \eta^i$ . If  $\eta \in \Lambda(M)$ , then  $\omega \wedge \eta$  is identified with  $f_i \otimes (\eta^i \wedge \eta)$ . So,  $\tilde{\mathcal{F}}(\omega \wedge \eta) = \mathcal{F}(f_i) \eta^i \wedge \eta = (\tilde{\mathcal{F}}(\omega)) \wedge \eta$ . This proves  $\Lambda(M)$ -linearity of  $\tilde{\mathcal{F}}$ .

Moreover,  $\bar{d}\omega = \overline{df_i \wedge \pi_M^*(\eta^i) + f_i \pi_M^*(d\eta^i)} = \bar{d}f \wedge \pi_M^*(\eta^i) + f_i \pi_M^*(d\eta^i)$  and  $\tilde{\mathcal{F}}(\bar{d}\omega) = (\tilde{\mathcal{F}} \circ \bar{d})(f_i) \wedge \eta^i + \mathcal{F}(f_i)(d\eta^i)$ . By commutativity of (36), the last expression equals to  $d(\mathcal{F}(f_i)) \wedge \eta^i + \mathcal{F}(f_i)(d\eta^i) = d(\mathcal{F}(f_i) \eta^i) = d(\tilde{\mathcal{F}}(\omega))$ .  $\square$

**Remark 10.** When  $\mathcal{F} = \int_{\mathcal{P}}$ , the identity  $\tilde{\mathcal{F}}(\bar{d}\omega) = d(\tilde{\mathcal{F}}(\omega))$  gives the familiar rule

$$(37) \quad \int_{\mathcal{P}} (d\omega_g) = d \left( \int_{\mathcal{P}} \omega_g \right)$$

for “taking the differential out of the sign of integration”.



## 9. THE HOMOTOPY FORMULA

The departing point to develop differential topology, which is the theory of algebraic topology based on differential forms, is the so-called *homotopy formula* (47), which allows to transform a homotopy connecting two maps from  $M$  to  $N$  into a chain homotopy connecting the induced maps from  $\Lambda(N)$  to  $\Lambda(M)$ .<sup>8</sup> Thanks to the theory of families of forms developed so far, we are able to show that the homotopy formula is a simple and natural consequence of the Cartan formula for the Lie derivative and the form-valued Newton–Leibniz formula (40). Our proof differs from the classical ones which can be found in literature (see the classical book [1]) in that, being purely algebraic, there is no need to check analytically the correctness of all the steps, thus focusing only its conceptual aspects.

Throughout this section,  $F : M \times \mathcal{P} \rightarrow N$  is a smooth homotopy,  $\omega$  is a form on  $N$ , and  $\eta = F^*(\omega)$ . So,  $\eta$  determines the horizontal form  $\bar{\eta}$ , which in its turn can be regarded as a  $\mathcal{P}$ -parametrized family of forms  $\{\eta_q\}$  (see Section 8 above). To distinguish the case when  $\mathcal{P}$  is  $\mathbb{R}$ , or an interval, we simply use the index  $t$  instead of  $q$ .

The derivative of the family  $\{\eta_q\}$  along a vector field  $X \in D(\mathcal{P})$  is defined by means of the Lie derivative  $L_{\tilde{X}}$  (see Corollary 3). Denote by  $\{X(\eta_q)\}$  the family of forms corresponding to  $L_{\tilde{X}}(\bar{\eta})$ .

A key remark is that the family  $\{X(\eta_q)\}$  is also obtained by slicing the form  $L_{X(F)}(\omega)$ , where  $\tilde{X}$  is the Lie derivative along the  $F$ -relative vector field  $X(F)$ . When  $X = \frac{d}{dt}$  we just write  $\frac{dF}{dt}$  instead of  $X(F)$ , and  $\eta'_t$  instead of  $X(\eta_t)$ .

To begin with, prove the form-valued Newton–Leibniz formula. Put for simplicity  $I_a^b = \int_a^b$  and  $\mathbb{I} = [a, b]$ .

**Lemma 7.**

$$(38) \quad I_a^b \circ \nabla_{\frac{\partial}{\partial t}} \circ p_0 = \iota_b^* - \iota_a^*.$$

*Proof.* By using decomposition (29) represent  $\omega$  in the form

$$(39) \quad \omega = \sum_i f_i \pi_M^*(\omega^i) + \rho \wedge \pi_{\mathbb{I}}^*(dt), \quad \omega^i \in \Lambda(M).$$

Then, since  $\overline{\rho \wedge \pi_{\mathbb{I}}^*(dt)} = 0$ , we have

$$\begin{aligned} \left( I_a^b \circ \nabla_{\frac{d}{dt}} \circ p_0 \right) (\omega) &= I_a^b \left( \nabla_{\frac{d}{dt}} \left( \overline{\sum_i f_i \pi_M^*(\omega^i) + \rho \wedge \pi_{\mathbb{I}}^*(dt)} \right) \right) \\ &= I_a^b \left( \nabla_{\frac{d}{dt}} \left( \sum_i f_i \pi_M^*(\omega^i) \right) \right) = I_a^b \left( \frac{\partial f_i}{\partial t} \pi_M^*(\omega^i) \right) \\ &= \sum_i \left( I_a^b \frac{\partial f_i}{\partial t} \right) \omega^i = \sum_i [(\iota_b^* - \iota_a^*)(f_i)] \omega^i \\ &= \sum_i (\iota_b^*(f_i) \omega^i - \iota_a^*(f_i) \omega^i). \end{aligned}$$

<sup>8</sup>It allows to prove homotopy invariance of the de Rham cohomology, the most fundamental property of the de Rham cohomology on which all efficient computational algorithms are based. Another basic instrument of computing de Rham cohomology, the *suspension theorem*, needs a compact-supported version of the theory of families, not discussed here.

On the other hand,

$$\begin{aligned}
(\iota_b^* - \iota_a^*)(\omega) &= (\iota_b^* - \iota_a^*) \left( \sum_i f_i \pi_M^*(\omega^i) + \rho \wedge \pi_{\mathbb{I}}^*(dt) \right) \\
&= \iota_b^* \left( \sum_i f_i \pi_M^*(\omega^i) + \rho \wedge \pi_{\mathbb{I}}^*(dt) \right) - \iota_a^* \left( \sum_i f_i \pi_M^*(\omega^i) + \rho \wedge \pi_{\mathbb{I}}^*(dt) \right) \\
&= \sum_i (\iota_b^*(f_i) \omega^i - \iota_a^*(f_i) \omega^i).
\end{aligned}$$

□

**Corollary 9.** *Let  $\omega \in \Lambda(M \times [a, b])$ . Then*

$$(40) \quad \int_a^b \omega'_t = \omega_b - \omega_a.$$

*Proof.* First observe that  $\omega_b - \omega_a = (\iota_b^* - \iota_a^*)(\omega)$ . On the other hand, the family  $\{\omega'_t\}$  corresponds to the differential form  $L_{\frac{d}{dt}}(\omega)$  (see Corollary 3) and hence

$$(41) \quad \int_a^b \omega'_t = \mathbf{I}_a^b(\overline{L_{\frac{d}{dt}}(\omega)}) = (\mathbf{I}_a^b \circ p_0)(\overline{L_{\frac{d}{dt}}(\omega)}) = (\mathbf{I}_a^b \circ p_0 \circ L_{\frac{d}{dt}})(\omega),$$

Moreover,

$$(42) \quad p_0 \circ L_{\frac{\partial}{\partial t}} = \nabla_{\frac{\partial}{\partial t}} \circ p_0.$$

Indeed, left and right hand sides operators restricted to horizontal forms coincide with  $\nabla_{\frac{\partial}{\partial t}}$ . Also, these operators annihilate vertical differential forms, since  $L_{\frac{\partial}{\partial t}}$  preserves the class of vertical forms (see Proposition 5) while  $p_0$  annihilates it.

Now it follows from (41) and Lemma 7 that

$$(43) \quad \int_a^b \omega'_t = (\mathbf{I}_a^b \circ p_0 \circ L_{\frac{d}{dt}})(\omega) = (\mathbf{I}_a^b \circ \nabla_{\frac{\partial}{\partial t}} \circ p_0)(\omega) = (\iota_b^* - \iota_a^*)(\omega) = \omega_b - \omega_a$$

□

Formula (40) is a generalization of the historical Newton–Leibniz formula, to smooth homotopies and differential forms. We call it “universal”, since the Newton–Leibniz formula for a particular homotopy  $F$  is derived from it by means of  $F^*$ .

Let now  $F : M \times \mathcal{P} \rightarrow N$  be a smooth homotopy, and  $\eta = F^*(\omega)$ .

**Corollary 10.** *It holds*

$$(44) \quad \int_a^b \eta'_t = \eta_b - \eta_a.$$

*Proof.* A particular case of (40), where  $\omega$  is replaced by  $F^*(\omega)$ . □

**Remark 11.** *Notice that (44) may be read as  $(\mathbf{I}_a^b \circ \nabla_{\frac{\partial}{\partial t}} \circ p_0 \circ F^*)(\omega) = (F_b^* - F_a^*)(\omega)$ . Since  $\omega$  is arbitrary, this implies*

$$(45) \quad \mathbf{I}_a^b \circ \nabla_{\frac{\partial}{\partial t}} \circ p_0 \circ F^* = F_b^* - F_a^*.$$

*In its turn, (45) is obtained from (38), by composing on the right the latter with  $F^*$ . This shows the universality of (38).*

Let  $\{\eta_t^F\}$  be the family of forms on  $M$  determined by  $i_{\frac{d}{dt}F}(\omega) = i_{\frac{d}{dt}}(F^*(\omega))$ . The operator

$$h^F : \Lambda(N) \rightarrow \Lambda(M), \quad \omega \mapsto \int_a^b \eta_t^F$$

is called the *homotopy operator* associated with the homotopy  $F$ . Equivalently,

$$(46) \quad h^F = \mathbf{I}_a^b \circ p_0 \circ i_{\frac{d}{dt}F}.$$

Obviously,  $h^F$  is a linear operator of degree  $-1$ .

**Theorem 1.** *The following homotopy formula takes place.*

$$(47) \quad F_b^* - F_a^* = [h^F, d].$$

*Proof.* By combining (45) and (42) we have

$$(48) \quad \begin{aligned} F_b^* - F_a^* &= \mathbf{I}_a^b \circ \nabla_{\frac{d}{dt}} \circ p_0 \circ F^* = \mathbf{I}_a^b \circ p_0 \circ L_{\frac{d}{dt}} \circ F^* \\ &= \mathbf{I}_a^b \circ p_0 \circ \left( i_{\frac{d}{dt}} \circ d \circ F^* + d \circ i_{\frac{d}{dt}} \circ F^* \right) \\ &= \mathbf{I}_a^b \circ p_0 \circ \left( i_{\frac{d}{dt}} \circ F^* \circ d_N + d_{M \times [a,b]} \circ i_{\frac{d}{dt}} \circ F^* \right) \\ &= \mathbf{I}_a^b \circ p_0 \circ i_{\frac{d}{dt}F} \circ d_N + \mathbf{I}_a^b \circ p_0 \circ d_{M \times [a,b]} \circ i_{\frac{d}{dt}F}. \end{aligned}$$

On the other hand Corollary 8 shows that the composition

$$\Lambda^*(M \times [a, b]) \xrightarrow{p_0} \Lambda_{\pi_M}^*(M \times [a, b]) \xrightarrow{\mathbf{I}_a^b} \Lambda^*(M)$$

is a cochain map, i.e.,  $\mathbf{I}_a^b \circ p_0 \circ d_{M \times [a,b]} = d_M \circ \mathbf{I}_a^b \circ p_0$ . This fact allows to rewrite formula (48) as

$$F_b^* - F_a^* = h^F \circ d_N + d_M \circ h^F.$$

□

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